Directed paths on hierarchical lattices with random sign weights

Eduardo G. Aponte
Laboratorio de Física Computacional, Universidad Nacional Experimental Politécnica, Torre Domus 14-B, Caracas, Venezuela

Ernesto Medina
Laboratorio de Física Estadística de Sistemas Desordenados, Centro de Física, Instituto Venezulano de Investigaciones Científicas, Apartado 21827, Caracas 1020A, Venezuela

(Received 15 April 1998)

We study sums of directed paths on a hierarchical lattice where each bond has either a positive or negative sign with a probability p. Such path sums J have been used to model interference effects by hopping electrons in the strongly localized regime. The advantage of hierarchical lattices is that they include path crossings, ignored by mean field approaches, while still permitting analytical treatment. Here we perform a scaling analysis of the controversial ''sign transition'' using Monte Carlo sampling, and conclude that the transition exists and is second order. Furthermore, we make use of exact moment recursion relations to find that the moments $\langle J^n \rangle$ always determine, uniquely, the probability distribution $P(J)$. We also derive, exactly, the moment behavior as a function of p in the thermodynamic limit. Extrapolations $(n \to 0)$ to obtain $\langle \ln J \rangle$ for odd and even moments yield a new signal for the transition that coincides with Monte Carlo simulations. Analysis of high moments yield interesting ''solitonic'' structures that propagate as a function of p. Finally, we derive the exact probability distribution for path sums J up to length $L=64$ for all sign probabilities.
[S1063-651X(98)04610-8]

PACS number(s): 02.50.Ng, 64.60.Cn, 72.20.−i

I. INTRODUCTION

Sums of directed paths are present in numerous models of disordered systems. Polymer configurations in a disordered matrix, dynamics of interfaces grown by deposition [1] and Feynman path sums for electron hopping between impurities [2,3] are only a few of the relevant examples. In this paper, we focus on the latter example, involving a model first introduced by Nguyen, Spivak, and Shklovskii (NSS) for interference effects in the strongly localized regime [2].

In the directed path sign model one studies the sum of all possible directed paths between two sites on a lattice. On each lattice bond, one places a random sign with probability p. Each directed path evolved is then computed by multiplying the values of the bonds it crosses. Finally the sum J of all paths is obtained. The proponents of the model [2] obtained, numerically for small systems, that a second order transition occurred at $p_c \sim 0.05$ between a phase with preferential sign (for the path sum J), and a phase with no preferential sign. NSS also offered appealing arguments based on the behavior of $\delta J/\langle J \rangle$. Presumably, such a parameter grows exponentially above the transition, while it goes to zero below p_c. The physical relevance of this transition lies in the fact that it may signal the change between Aharonov-Bohm oscillations of period hc/e and those of $hc/2e$ [2] in the context of hopping conduction.

The NSS argument was later contended by Shapir and Wang [4], arguing that correlations between paths implied that $\delta J/\langle J \rangle$ does not necessarily go to zero for any p. Subsequently, Wang *et al.* [5] used an exact enumeration scheme to probe the transition for small lattices of maximum size $L=9$. The work found no evidence of a transition above negative sign probability $p=0.02$. Such conclusions were supported by Zhao *et al.* [6] on the basis of numerics, for large square lattices, where it was assumed that the transition did not exist above $p=0.025$ in two dimensions. Nevertheless, the decay of the order parameter ΔP as a function of system size was found to be anomalously slow for finite p (see also Ref. [3]). Thus, more recently, Spivak, Feng, and Zeng [7] discussed numerical results that suggest a finite jump in the order parameter indicating a first order transition for the sign problem. The authors also implied that the moments $\langle J^n \rangle$ increase faster than $n!$ as $n \to \infty$, indicating there is no unique relation between $\langle J^n \rangle$ and the probability distribution $P(J)$. This is an important point since the moments, in such a case, may not contain information about the transition. Finally, in a recent paper by Nguyen and Gamietea [8], a renewed extensive study of the parameter $\delta J/\langle J \rangle$ proves that, at least according to such a parameter, no transition exists; only a strong crossover from logarithmic to exponential behavior is observed.

Besides the numerical approaches, mean field type approximations by Obukhov [9] point to a second order transition for dimensions $d \geq 4$. Furthermore, Derrida and Cook [10] also took up the problem, analytically, using a sparse matrix approach. They generalized the model to random phases, which includes random signs as a special case. Their approach is mean field in nature, and results in a phase diagram where the sign transition is of second order [11] (see also Ref. [12]). Nevertheless, mean field results may not apply to lower dimensions due to the importance of path crossings [4].

Here we address the following issues: (i) What is the order of the sign transition through a scaling analysis of the order parameter proposed? (ii) Do moments of the path sums determine the probability distribution uniquely? (iii) What is the exact behavior of the parameter $\delta J/\langle J \rangle$ above and below the transition? An interesting perspective will be gained by using a hierarchical lattice: Such lattices, while still amenable to analytical manipulation, include crucial path correlation effects absent in the mean field.

The paper is organized as follows: Section II discusses the sign model and describes hierarchical lattices. In Sec. III, we perform detailed Monte Carlo simulations, close to the transition, for systems up to size $L=512$. A scaling analysis is performed for the order parameter $\Delta P=P(J>0)-P(J<0)$ to distinguish between first and second order transitions. In Sec. IV we study the moments $\langle J^n\rangle$ exactly, using moment recursion relations [13]. We find that moments determine the distribution uniquely according to Carlemans theorem, and find possible indications of a phase transition from odd and even moment extrapolations to $n=0$. In this section we also discuss the high moment behavior, unveiling interesting structures as a function of the sign probability p. Subsequently, we probe the parameter $\delta J/\langle J\rangle$ exactly, showing its unambiguous crossover between exponential and logarithmic behavior. In Sec. V we obtain the exact probability distribution for lattice sizes $L=16$, and sample the distribution for up to $L=64$ as a function of p. We end with the conclusions and a discussion of the mapping of the moments to an n-body partition function in one dimension as a continuum model that might aid in explaining the curious high moment behavior.

II. SIGN MODEL

Imagine two reference points on a lattice between which one would like to evolve all possible *directed paths* and compute a ''partition function''

$$J=\sum_i \Gamma_i, \tag{1}$$

where Γ_i represents each individual contributing path. By directed it is meant that paths always propagate in the forward direction without loops or overhangs. The random medium in which these paths evolve can be represented by assigning local weights [14] on the bonds or sites that are picked up by the paths as they wander to their final destination. Such a model has been used as a paradigm simulating, for example, a coarse-grained polymer or interface wandering in a random matrix with locally favorable energy minima [14]. The model is interesting because it yields anomalous lateral wandering and energy exponents for the interface/polymer as compared to those generated by simple diffusion, signaling a new disorder-induced universality class in $(1+1)$ dimensions.

Another application, in an entirely different field, is in the context of variable range hopping [15], a mechanism for conduction in insulators. In this context, one also needs to sum over Feynman paths to compute the transition probability, between impurities, of current bearing electrons. The Feynman paths, in this case, are directed because they are tunneling paths. Any elongation of the latter, in the form of loops or overhangs, is exponentially less probable. For further justification of the model we refer the reader to the review in reference [3]. NSS studied such tunneling processes, and proposed a directed path model where the local weights are random signs [2]. In such a model, the path Γ_i is a product of the signs it picks up en route to the final site. Writing Eq. (1) more explicitly,

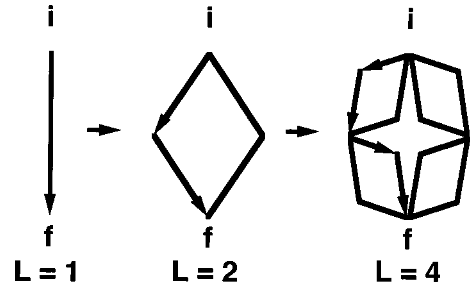

FIG. 1. Hierarchical lattices are built by repeating a chosen motif; each bond turns into a diamond recursively. The figure shows successive iterations of the lattice and the corresponding length L between end points i and f. Examples of a directed path at each order are indicated by contiguous arrows.

$$J=\sum_{\Gamma_i}\prod_i \eta_i, \tag{2}$$

where η_i is a random sign according to the distribution $P(\eta)=p\,\delta(\eta-1)+(1-p)\,\delta(\eta+1)$. The probability p in the NSS model emulates the relative abundance of levels above and below the Fermi energy [2]. This model has been very successful in explaining qualitative and quantitative features of conduction in the strongly localized regime. In particular, intriguing interference effects producing a characteristic periodicity of magnetic field oscillations [16] and changes in the localization length due to nonlocal effects [3,17]. In spite of the seemingly different nature of disorder in the NSS model, replica arguments and numerics have shown that it belongs to the same universality class of directed polymers with positive weights [3,18], at least for p close to $\frac{1}{2}$.

We have taken up the sign model on hierarchical lattices, as mentioned in Sec. I. A hierarchical lattice is a recursive structure built by repeating a chosen motif [19]. Depending on the latter motif, one can build integer dimensional objects emulating Euclidean lattice. For this work we chose the Berker lattice or diamond. Such a motif (see Fig. 1) has a parameter b corresponding to the number of branches between the initial i and final f points. The lattice size is related to the recursion order m as $L=2^{m-1}$, i.e., the number of bonds on any directed path between i and f. The number of bonds on the lattice (or mass) is given by $M=(2b)^{m-1}$, so that the effective dimension of the lattice is $d_{\text{eff}}=1+(\log b/\log 2)$. In this work we will use $b=2$ except if otherwise stated. Qualitative features of critical behavior of many statistical models are correctly reproduced on such structures with no unphysical effects. In fact, mapping to hierarchical lattices is the basis of the Migdal-Kadanoff renormalization procedure, of frequent use in critical phenomena. As noted above, an important advantage of hierarchical lattices over either Bethe lattices/mean field approaches is that path intersections are taken into account. Thus we expect that the resulting simulations will be more faithful to low dimensional behavior. In fact, we will present, in Sec. IV, further evidence of the adequacy of hierarchical lattices making contact with known recent results on the sign transition.

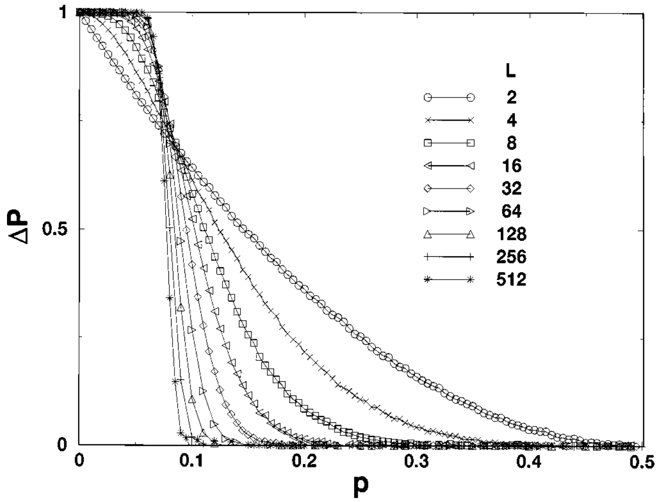

FIG. 2. The figure shows the order parameter $\Delta P = P(J>0) - P(J<0)$ as a function of the sign probability p for system sizes indicated. Averages were performed over more than 20 000 realizations of randomness. Note the formation of a plateau at $\Delta P = 1$ for small p.

III. SIGN PHASE TRANSITION IN TWO DIMENSIONS

In this section, we have undertaken Monte Carlo simulations on hierarchical lattices to check for scaling properties. Paradoxically, scaling has only been discussed once before in connection with the transition [18], and it is a primary tool to assess its nature. It will be especially useful to clearly distinguish between first order and second order transitions.

Hierarchical lattices were generated to $L=512$ or order 10. Averages were taken over 20 000 realizations of disorder for a series of p values between 0 and 0.5. As the size of the system increases, more detailed data were collected close to the transition regime $0.05<p<0.1$. Figure 2 shows Monte Carlo data for the order parameter ΔP as a function of p. A definite plateau at $\Delta P=1$ develops as L increases for low p, signaling a definite change in the order parameter (positively signed paths dominate).

For the proposed order parameter we should expect the scaling form $\Delta P = f((p-p_c)L^{1/\nu})$. Figure 3 shows a good collapse for the same data as the previous figure. As the order parameter is always between 0 and 1, we only need to find p_c and the correlation length exponent ν. For the hypothetical transition we find the values $p_c = 0.071 \pm 0.001$ and $\nu = 1.85 \pm 0.07$ $(1/\nu = 0.54)$. The latter exponent is very different from that of percolation on these lattices $\nu = \ln 2/(\ln 2 + \ln(3-\sqrt{5})) = 1.63529...$; so the role of percolation, if any, is not apparent. If the transition were first order the exponent $1/\nu$ would be the dimensionality of the system d [20]. The nontrivial scaling found can also be seen by taking the derivative of the order parameter and plotting its maximum as a function of the system size. These criteria rule out a first order transition.

We have also monitored the evolution of $p_c(L)$ with size. The specific value of $p_c(L)$ was found from the peak values of the derivative of the order parameter ΔP. The resulting values are plotted in Fig. 4, where, within error bars, the values of $1/\nu$ and $p_c(\infty)$ are confirmed. Summarizing, scaling is very good around $p_c = 0.07$, and *does not* correspond to the scaling of a first order transition. Furthermore, there is no sign of a discontinuity in the order parameter, as suggested in Ref. [7]. We thus conclude that, on hierarchical lattices, the transition exists and is second order as mean field predicts. These conclusions are in agreement with work by Roux and Coniglio [18] on hierarchical lattices. There they analyzed the variable $\alpha_i = (n_i^+ - n_i^-)$, where $n_i^\pm$ is the fraction of positive (negative) paths arriving at site i, and they suggested a clear positive α phase. The order of the transition for hierarchical lattices was not analyzed in detail in their paper. Nevertheless, they noted an undue emphasis of hierarchical lattices on the $\alpha=0$ result, and the possible impact of this on the scaling properties of various quantities. We will come back to such observations, briefly, in Sec. V.

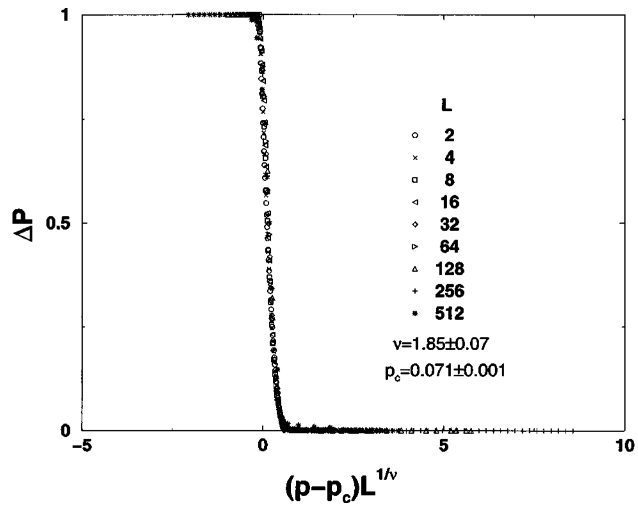

FIG. 3. Same data as in Fig. 2 after collapsing the curves for different system sizes. The appropriate choices for p_c and ν, the transition threshold and the correlation length exponent, are indicated.

IV. MOMENT RECURSION RELATIONS

A statistic we can probe exactly on hierarchical lattices are the moments of the probability distribution. This is pos-

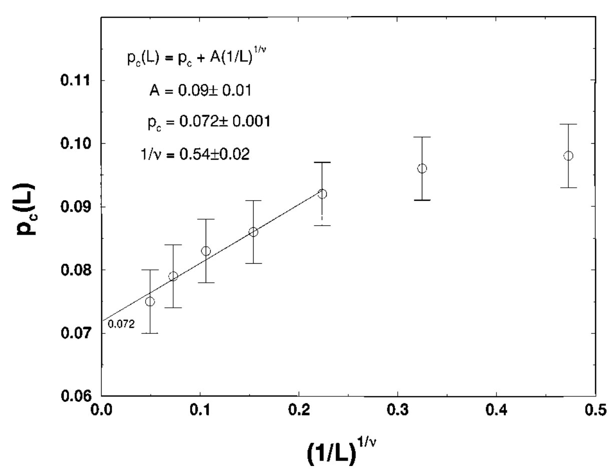

FIG. 4. The figure shows the value of $p_c(L)$, evaluated from the peaks of the derivative of the order parameter, as a function of $(1/L)^{1/\nu}$. The last five sizes from $L=32-512$ have been fitted by a least squares method to yield the asymptotic value $p_c(\infty) = 0.072$ indicated.

sible because of recursion relations derived by Cook and Derrida [21], and generalized to arbitrary moment and hierarchical order (system size) by Medina and Kardar [13]. The recursion relation for $b=2$ is

$$\langle J_{m+1}^{n}\rangle=\sum_{s=0}^{n}\frac{n!}{(n-s)!s!}[\langle J_m^s\rangle]^2[\langle J_m^{n-s}\rangle]^2, \tag{3}$$

where n is the moment number and m is the hierarchical lattice order. This expression is readily generalized to other integer b by changing the binomial factor to a multinomial, and including the additional branches. Hence one can emulate higher-dimensional networks. The simple form of this recursion permits one, given the local moments at order 1, to compute moments to any given lattice size. Appropriate programming of the recursion relations, with arbitrary precision computations, is linear in time with lattice order.

The behavior of the moments for the sign model is extremely rich, as we shall see in the following. As found in Ref. [13], after a few hierarchical orders, the values $\ln(\langle J_m^n\rangle)/L$ converge rapidly to a limiting form as a function of n. Such limiting form is important because it also signals the convergence to a unique limiting distribution, at least if moments do not grow faster than $n!$ [22]. The asymptotic form of the moments can be obtained for $p=0$ [13,21],

$$\frac{\ln\langle J^n\rangle}{L}=n\left(1-\frac{1}{L}\right)\ln 2; \tag{4}$$

that is, moments grow exponentially with n for $p=0$. Nevertheless, for $0<p<\frac{1}{2}$, lower moments grow slightly faster than exponentially [$\exp(n^\alpha)$, with $1<\alpha<2$], gradually converging to exponential growth for larger moments. The latter implies, according to the condition

$$\sum_{n=0}^{\infty}\langle J^{2n}\rangle^{-1/2n}=\infty, \tag{5}$$

that the moments determine the distribution uniquely. There are various forms of such a theorem, but the above is the strongest version due to Carleman [22]. If one substitutes $\langle J^{2n}\rangle\sim\exp(2n)$—our asymptotic result—above, the criterion is satisfied. Even if $\langle J^{2n}\rangle$ grows slightly faster i.e., $\exp(2n\ln 2n)\sim(2n)!$, the above sum diverges because $\Sigma_n 1/n=\infty$. Any faster growth would violate Eq. (5), factorial growth being the borderline case. That the moments $\langle J^n\rangle$ satisfy Eq. (5) is one of our central results. In Fig. 5 we show a sequence of moments as a function of the moment number n. The different curves, starting from below, represent hierarchical orders 1–9 (sizes $L=2-256$). One readily notes convergence to a definite law. The inset shows a comparison between the growth of $n!$ and that of moments for the particular case of $p=0.1$. The asymptotic behavior is already reached at $L=128$, larger sizes falling on the same curve.

For values close to $p=\frac{1}{2}$, the moment sequence has a characteristic sawtooth shape, where even moments are at the crests and the odd at the troughs. Such structure is not a finite size effect. We have checked this for up to $L=2^{20}$ on the hierarchical lattice. As $p\rightarrow\frac{1}{2}$ all the odd moments go to zero, while the even remain finite, as expected. On the other hand, as p is reduced the sawtooth disappears, first for the higher moments and then for the lower. In this respect there appears to be a phase transition for each moment at different values of p, in a way reminiscent of that discussed by Cook and Derrida [21] (in their case as a function of ‘‘temperature’’). The transition for the first two moments occurs close to $p=0.075$, which is close to that found from Monte Carlo simulations in Sec. III. On this basis it is plausible that the disappearance of sawtooth shape is related to the transition.

Figure 6 shows a set of curves for $d\ln(\langle J_m^n\rangle/L)/dn$, and various values of p and $L=2^{17}$ up to $n=100$. The last six orders of the hierarchical lattice collapse on the same curve, indicating that we have achieved asymptotics. For the high-

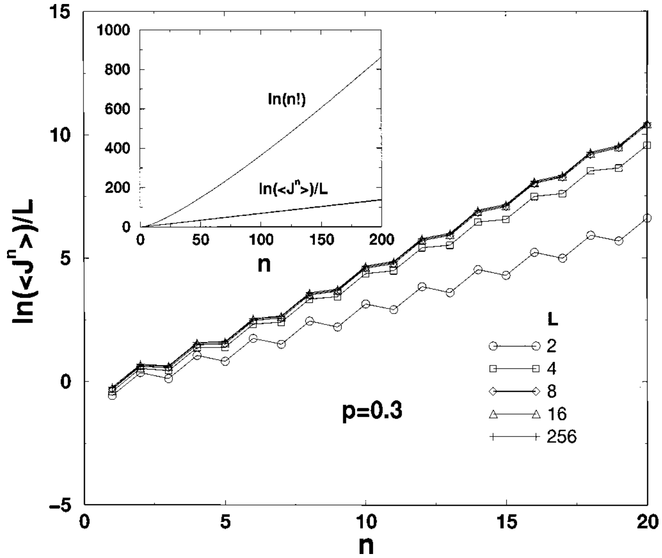

FIG. 5. The moments $\ln\langle J^n\rangle/L$ as a function of the moment number n for the lattice sizes indicated. The figure shows the rapid convergence to an asymptotic result. In the inset, we show that while the initial moments grow faster than exponential they nevertheless grow slower than $n!$, so there is a unique relation between moments and probability distribution.

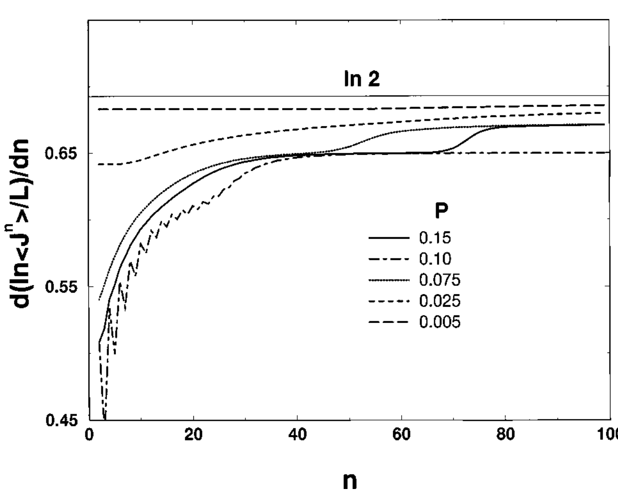

FIG. 6. The derivative of $\ln\langle J^n\rangle/L$ as a function of the moment number, for size $L=2^{15}$ (last six orders collapse onto the same curves.) As p decreases the curves approach the asymptotic value ln 2. Note the change from the sawtooth behavior above $p=0.07$ to collinearity. Shoulder features, developing at higher n, move almost undeformed in the positive n direction as p increases.

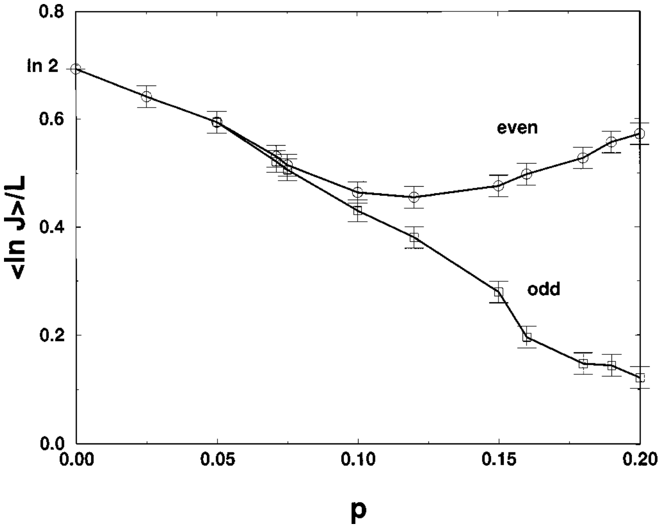

FIG. 7. The ''free energy'' $\langle \ln J\rangle/L$, as discussed in the text, as a function of p. Note that the curves extrapolated from even and odd moments merge around the threshold for the transition obtained from Monte Carlo.

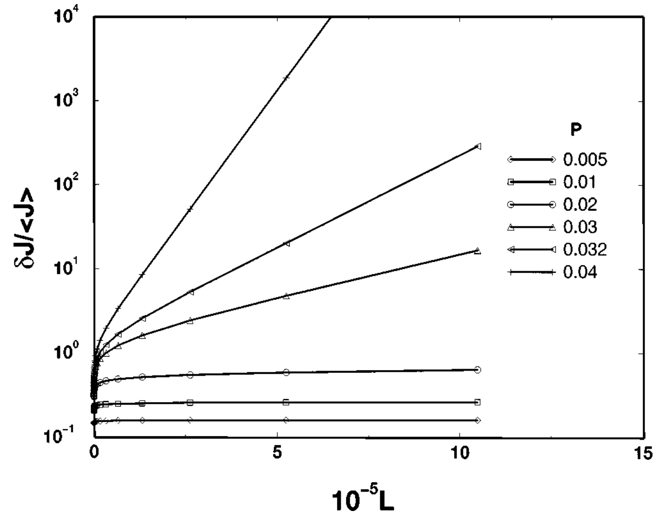

FIG. 8. The figure shows the behavior of $\delta J/\langle J\rangle$ as a function of L for the p values indicated. As the plot is semilog the exponential behavior above $p=0.03$ is evident.

est value of p one notes the sawtooth behavior, while it disappears for all moments below $p=0.1$. Nevertheless, additional structure is observed at moments beyond $n=40$ for $p=0.075$ and $p=0.1$, where a shoulder develops and moves toward larger n values as p increases, undeformed, in a solitonic manner. Although the analysis of these structures is beyond the scope of this paper, it is interesting to analyze it in the light of a mapping to a one-dimensional many body problem [23]. In such a mapping the moment number corresponds to the number of particles interacting like charges on contact. Thus we speculate that the shoulders could be related to sudden changes in the character of the ground state as the particle number (moment number) increases. We will discuss this in more detail in Sec. VI.

For even smaller p values the curve starts to resemble the well known $p=0$ limit given by Eq. (4), and depicted as a flat line at ln 2 in Fig. 6. From the figure one can graphically identify the value of $\langle \ln J\rangle$ as a function of p using the relation $d\ln\langle J_m^n\rangle/dn|_{n\to 0}=\langle \ln J\rangle$. The quantity $\langle \ln J\rangle$ is a ''free energy'' that may reflect the sign transition. We have followed the value at intercept mentioned before as a function of p below $p=0.2$. When the moments ''zigzag'' there are two possible extrapolations, while below the assumed transition the moments lead to a single prediction of the free energy. The results are depicted in Fig. 7, where the curves merge around $p_c=0.07$ within the error of the extrapolation procedure. Such a value coincides with our Monte Carlo prediction.

One can validate the relevance of hierarchical lattices by checking the exact computation of the variable $\delta J/\langle J\rangle$ with $\delta J=\sqrt{\langle J^2\rangle-\langle J\rangle^2}$. Such a quantity was discussed extensively in previous work [2,4,8,24]. As mentioned before, $\delta J/\langle J\rangle$ was initially suggested as a candidate for a kind of order parameter that diverged exponentially above the transition and went to zero below. Observations by Shapir and Wang [4] showed, nevertheless, that path correlations (crossings) invalidated the vanishing of the parameter for any value of p. It has been argued that for small p there is a crossover from exponential growth (for $p>p_c$) to logarithmic growth (for $p<p_c$) [8,24]. Shapir and Wang, on the other hand, found a change from $\exp[|\ln(1-2p)|2a\sqrt{L}]$ for $p<p_c$ to $\exp[|\ln(2(1-2p)^2)|2L]$ for $p>p_c$. However, they observed that the former result is incorrect because partial overlaps of pairs of walks should be accounted for.

Simulations on regular lattices to date can only do very poorly in proving the surmized logarithmic behavior below p_c. Here we have computed $\delta J/\langle J\rangle$ to sizes $L=2^{20}$ for various p values in a few CPU minutes. We have found a clear confirmation of logarithmic to exponential crossover as p increases. Figures 8 and 9 show $\delta J/\langle J\rangle$ and its derivative as a function of L, respectively. The scales used permit rapid identification of the corresponding behavior. It should be noted that, on Euclidean lattices, the reported behavior is $\delta J/\langle J\rangle \propto (\ln L)^{\mu}$, where $\mu\sim 1$ but depends weakly on p.

On hierarchical lattices we can also demonstrate analytically that there is no transition in the variable $\delta J/\langle J\rangle$. Fol-

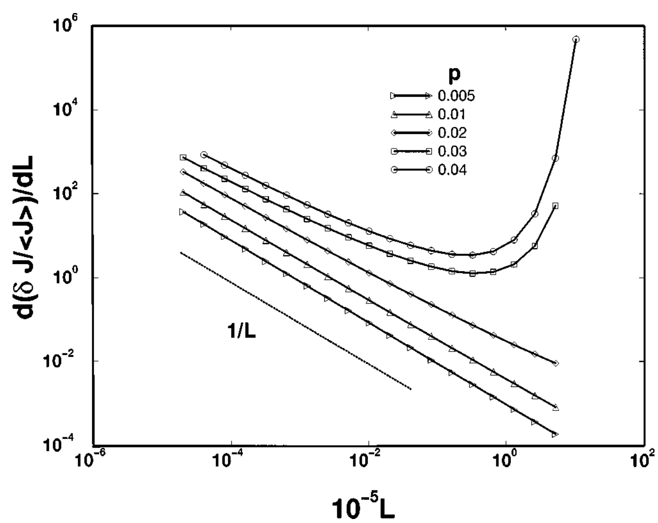

FIG. 9. The figure shows the derivative of the data in Fig. 8. Here the logarithmic behavior of $\delta J/\langle J\rangle$ is evident. The dotted line is a guide for the eye for $1/L$ behavior.

lowing Cook and Derrida [21], Eq. (3); for the first two moments, can be written for general b as

$$\langle J_{m+1}\rangle = b\langle J_m\rangle^2,$$
$$\langle J_{m+1}^2\rangle = b\langle J_m^2\rangle^2 + b(b-1)\langle J_m\rangle^4. \tag{6}$$

Now, after defining $j_2(m)=\langle J_m\rangle^2/\langle J_m^2\rangle$ one can write a recursion relation for $\delta J/\langle J\rangle$ as

$$\left(\frac{\delta J_{m+1}}{\langle J_{m+1}\rangle}\right)^2 = \frac{1}{b}\left[\frac{1-j_2^2(m)}{j_2^2(m)}\right]. \tag{7}$$

It is simple to determine that j_2 has in general three fixed points: $j_2=0$, 1, and $1/(b-1)$. For $b>2$ ($d_{eff}>2$) a critical fixed point arises, and $\delta J/\langle J\rangle$ exhibits a phase transition as NSS proposed. On the other hand, for $b=2$ there are only two trivial fixed points; $j_2=1$ is unstable and $j_2=0$ is stable, indicating that $\delta J/\langle J\rangle$ always diverges as found above. Values of j_2 close to one correspond to $p\rightarrow 0$, while j_2 close to zero correspond to $p\rightarrow \frac{1}{2}$. Analyzing the behavior of the recursion for j_2 near the $j_2=0$ fixed point, one can derive from Eq. (6) that $\delta J/\langle J\rangle \sim \frac{1}{2}\exp[L(|\ln j_2(0)|+\frac{1}{2}j_2^2(0))]$. The behavior close to $j_2=1$, which should be logarithmic, is also verified (numerically), although we have not arrived at a simple closed expression. In summary, hierarchical lattices provide similar results to those expected on Euclidean lattices, thus seeming a good testing ground for the sign transition.

As a final word, we have computed higher order cummulants of J, finding no features of special interest related to the transition. The only result worth mentioning is that $\ln(C_j)^{1/j}/L=\ln 2$ for $p=0$, where C_j is the jth cummulant of J. In what follows we will take advantage of the special structure of hierarchical lattices to compute the full probability distribution for J.

V. PROBABILITY DISTRIBUTION FOR *J*

Monte Carlo sampling of the distribution of J is handicapped by the models' distribution broadness. For such reasons, Wang *et al.* [5] undertook an exact enumeration study to probe the NSS order parameter $\Delta P=P(J>0)-P(J<0)$. Because of the high computer demand of exact enumeration, they could only access sizes of $L=10$ for all p. Here we use a scheme, on hierarchical lattices, permitting access to $L=16$ exactly for all p and a sampling of the distribution for $L=64$. The procedure is as follows: As a hierarchical lattice is built recursively following a chosen motif, one can write the following recursion relation for the probability distribution:

$$P_{m+1}(J)=\prod_{i=1}^{4}\int_{-\infty}^{\infty}P_m(\eta_i)\,\delta(J-\eta_1\eta_2-\eta_3\eta_4)\,d\eta_i\,, \tag{8}$$

where $\eta_{1,2}$ and $\eta_{3,4}$ denote contiguous elements on separate branches of the hierarchical lattice. $P_1=p\,\delta(\eta-1)+(1-p)\,\delta(\eta+1)$, where p is the sign probability discussed in previous sections. The number of possible outcomes for J or number of different paths goes as $2^{2^{m-1}-1}$ (32 768 for $L=16$ and $m=5$, and 2 147 483 648 for $L=32$ and $m=6$). This growth is extremely fast, although many J values will be degenerate for any particular disorder realization. Note that while $L=16$ is easily accessible, going an order further puts the calculation out of reach, no intermediate sizes being available on hierarchical lattices. For $L=32$ we have resorted to a coarse-graining procedure in the following manner: the exact results for $L=16$ involve 175 terms which we cannot exactly evolve to the next order. Nevertheless, we can make a coarse-grained distribution by averaging J occurrences in groups of seven to obtain 25 different values. One can then go to up to $L=64$ by repeating this procedure. Beyond such a size, the coarse-graining procedure does not incorporate sufficient detail to see anymore changes in the distribution, so within our resolution we have achieved its limit form.

Figure 10(a) shows the probability distribution for $L=16$ for significant values of p. The probability distribution is astonishingly complex, even for small sizes, revealing rich interferences in the path sums J. Note that it would be hopeless to sample the distribution $P(J)$ using Monte Carlo as there are 60 to 130 orders of magnitude of probability. Figure 10(b) shows different p values for a sample $L=32$ as a function of p using the coarse-graining procedure described above. As expected, the distribution is symmetric for $p=0.5$ and gains asymmetry ($\Delta P\neq 0$) as p moves toward zero. Note that $P(J)$ falls more slowly than exponential, on average, about the peak value. The speed with which the distribution appears symmetric beyond $p=0.1$ is notable. This feature is understood in the ''zigzag'' behavior of the moments, where odd moments are much smaller than even moments and their separation increases exponentially as p is increased.

Having the information of the exact distribution one is also able to obtain the exact order parameter ΔP introduced in Sec. III. No qualitative differences were found with curves reported in Figs. 1 and 2, at least to sizes $L=32$, so sampling of ΔP, involved in Monte Carlo simulations, seems to be good enough to draw the conclusions about the transition (see Sec. III).

In Fig. 11 we have depicted the distribution for $L=64$ and $p=0.5$ without joining the points for the probability amplitude (as was done in Fig. 10). A fractal structure is apparent. The whole distribution, in the shape of an approximate triangle, is built from scaled, identical triangular structures up to the resolution achieved by the coarse-graining procedure. A similar complexity is expected for the sign problem on Euclidean lattices.

An interesting final point to make in this section is that, in view of the unique relation between distribution and moments (see Sec. IV), it is possible to use known inversion formulas [22]. In this way one could derive the limiting distribution exactly to any order desired.

VI. SUMMARY AND DISCUSSION

We have provided evidence of the existence of a phase transition for the directed path sign model on hierarchical lattices. Nontrivial finite size scaling of the order parameter close to the transition points to a second order phase transition as found from mean field type approaches. From numerical computations, the threshold on diamond hierarchical

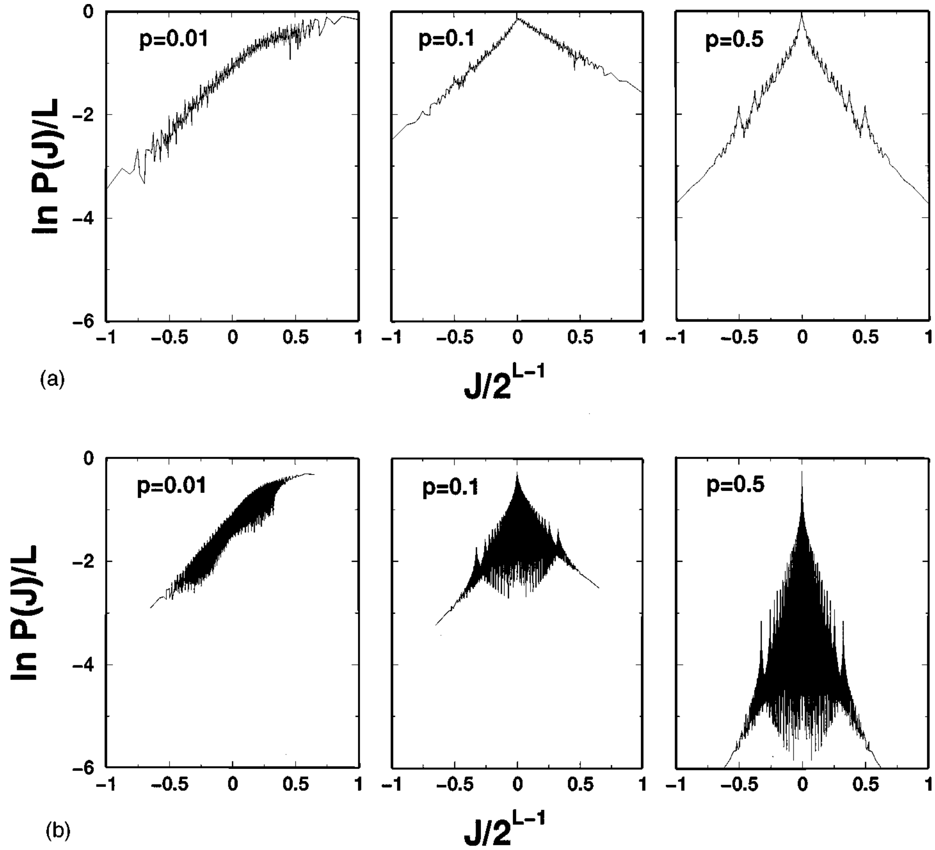

FIG. 10. (a) The exact probability distribution for $L=16$ at the p values indicated. Note the self-affine structure; the central peak is repeated at subsidiary local maxima. (b) The figure shows the coarse-grained distribution for $L=32$.

lattices is $p_c=0.071\pm0.001$, and the correlation length exponent is $\nu=1.85\pm0.07$. The latter exponent is very different from that of percolation on the same lattice $\nu=1.635...$.

The study of exact moment recursion relations for $\langle J^n\rangle$ led us to the definitive conclusion that the moments uniquely determine the probability distribution, according to Carleman's theorem [22]. Using extrapolations of the derivative of integer moments $(d\langle J\rangle^n/dn)$ to $n=0$, we were able to find a ''free energy'' $\langle \ln J\rangle$. Such a free energy splits into two possible extrapolations (from even and odd moments) as one goes through the transition point by increasing p. The latter transition point coincides with that found in Monte Carlo simulations of the sign transition. We have not completely interpreted this connection in the present paper. Furthermore, we studied the high moments of the partition function J below the transition, and found a very interesting nonmonotonic behavior including step structures that propagate on the moment number axis, as p changes.

Using the fact that moments can be computed exactly we studied the celebrated ratio $\delta J/\langle J\rangle$ proposed by NSS. We have shown, analytically, that indeed in $d_{\text{eff}}=2$ the ratio does not show a transition as suspected numerically [4,8,24] on regular lattices. Furthermore, we have shown that hierarchical lattices exhibit the same logarithmic to exponential crossover for $\delta J/\langle J\rangle$ surmized in Refs. [8] and [24].

Finally, we studied a recursion relation for the full probability distribution for J, finding an extremely complex structure even for systems as small as $L=16$. Previous remarks by Roux and Coniglio [18] of anomalous accumulation of probability at $J=0$ are confirmed. Nevertheless, their claim that the hierarchical lattice becomes essentially one dimensional for large L, and thus, that the probability distribution should approach a Gaussian, is not borne out from our results. One obvious difference is that, for a Gaussian, all cumulants larger than 2 should be 0, which is in disagree-

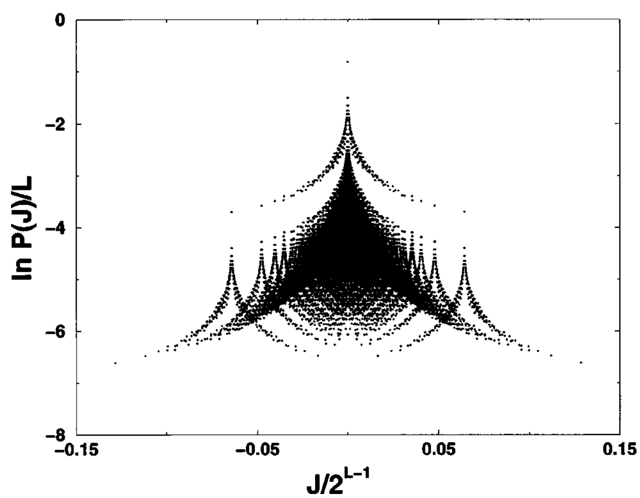

FIG. 11. The probability distribution for $L=64$ and $p=0.5$ using the coarse-grained distribution for $L=32$. To emphasize the self-similar structure we have not to joined the points in the graph as in Fig. 10.

ment with exact results of Sec. IV. No evident signal of the transition, beyond that already inferred from the order parameter ΔP, is found from the full probability distribution.

Medina and Kardar [3] studied the moments for the sign problem, interpreting them as partition functions for n-body one-dimensional Hamiltonians with contact interactions. Most of the focus, however, has been on the low n behavior that yields cummulants of $\ln J$. Nevertheless, it would be interesting to interpret the findings of this paper, regarding high moments, in the light of a many body theory. A previous effort by Zhang [23] focused on the Hartree-Fock approximation valid only for a large number of particles (higher moments). In Zhang's approach the sign model was equivalent to finding the ground state of the many body Hamiltonian

$$\left(-\sum_{i=1}^{2n} \partial_i^2 + \sum_{i>j} e_i e_j \delta(x_i - x_j)\right)\Psi(x_1 \cdots x_n) = E_0(n)\Psi(x_1 \cdots x_n), \tag{9}$$

where e_i is a charge that acts via contact interaction of the ith particle: $e_i = 1$ for $1 \leqslant i \leqslant n$ and $e_j = -1$ for $n \leqslant i \leqslant 2n$. Zhang's approach yielded $E_0 \propto n^2$. Our findings predict, from the relation $\ln\langle J^n\rangle/L = E_0$, $E_0 = \gamma n$ for large n (see also Ref. [3]), where γ increases as $p \to 0$. For lower n the behavior is nontrivial and is certainly not represented as a simple power law of n. Therefore, Zhang's results represent some kind of intermediate regime. A more detailed solution of Eq. (9) might yield the ''solitonic'' patterns reported here (see Sec. IV), which are not well understood. As speculated in Sec. IV, the ground state formed by particles with attractive and repulsive interactions might change, suddenly, at critical particle numbers generating discontinuities in the derivative of $\ln\langle J^n\rangle$. More work is needed in this direction.

The highly nonmonotonic behavior displayed by the moments calls for caution regarding the regime of validity of moments dependencies on the moment number n reported in the literature [3,23]. Claims of a nonunique relation between moments and the probability distribution [7] were based on expressions only valid in the $n \to 0$ limit, which is clearly unrelated to the constraints of Carleman's theorem [25]. Obviously, the conclusions of this paper are only valid in the measure to which hierarchical lattices agree with continuum results. For a discussion of the latter point, see Ref. [13].

ACKNOWLEDGMENTS

E.M. thanks G. Urbina and A. Hasmy for a careful reading of the manuscript, and R. Paredes for a valuable suggestion. E.G.A. thanks R. Rangel for useful discussions. This work was supported by CONICIT through Grant No. S1-97000368, a grant from the Research Fund of UNEXPO, and the POLAR Foundation.

[1] E. Medina, T. Hwa, M. Kardar, and Y.-C Zhang, Phys. Rev. A **39**, 3053 (1989).
[2] V. L. Nguyen, B. Z. Spivak, and B. I. Shklovskii, Pis'ma Zh. Éksp. Teor. Fiz. **41**, 35 (1985) [JETP Lett. **41**, 42 (1986)]; Zh. Éksp. Teor. Fiz. **89**, 11 (1985) [Sov. Phys. JETP **62**, 1021 (1985)].
[3] E. Medina and M. Kardar, Phys. Rev. B **46**, 9984 (1992).
[4] Y. Shapir and X. R. Wang, Europhys. Lett. **4**, 1165 (1987).
[5] X. R. Wang, Y. Shapir, E. Medina, and M. Kardar, Phys. Rev. B **42**, 4559 (1990).
[6] H. L. Zhao, B. Spivak, M. Gelfand, and S. Feng, Phys. Rev. B **44**, 10760 (1991).
[7] B. Spivak, S. Feng, and F. Zeng, Pis'ma Zh. Eksp. Teor. Fiz. **64**, 283 (1996) [JETP Lett. **64**, 312 (1996)].
[8] V. L. Nguyen and A. D. Gamietea, Phys. Rev. B **53**, 7932 (1996).
[9] S. Obukhov, in *Hopping Conduction in Semiconductors*, edited by M. Pollak and B. I. Shklovskii (North-Holland, Amsterdam, 1990), p. 338.
[10] J. Cook and B. Derrida, J. Stat. Phys. **61**, 961 (1990).
[11] In Ref. [10], Cook and Derrida add a temperature parameter. The limit $T \to \infty$ corresponds to the pure sign model we treat here.
[12] Y. Y. Goldschmidt and T. Blum, J. Phys. I **2**, 1607 (1992).
[13] E. Medina and M. Kardar, J. Stat. Phys. **71**, 967 (1993).
[14] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[15] B. I. Shklovskii and A. L. Efros, *Electronic Properties of Doped Semiconductors* (Springer-Verlag, Berlin, 1984).
[16] B. I. Shklovskii and B. Z. Spivak, in *Hopping Conduction in Semiconductors* (Ref. [9]).
[17] F. P. Milliken and Z. Ovadyahu, Phys. Rev. Lett. **65**, 911 (1990).
[18] S. Roux and A. Coniglio, J. Phys. A **27**, 5467 (1994).
[19] A. N. Berker and S. Ostlund, J. Phys. C **12**, 4961 (1979).
[20] J. Lee and J. M. Kosterlitz, Phys. Rev. B **43**, 3265 (1991).
[21] J. Cook and B. Derrida, J. Stat. Phys. **57**, 89 (1989).
[22] W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1971).
[23] Y.-C. Zhang, J. Stat. Phys. **57**, 1123 (1989).
[24] B. I. Shklovskii and B. Z. Spivak, J. Stat. Phys. **38**, 267 (1985).
[25] E. Medina (unpublished).